Complexity of finding efficient allocations of highest welfare

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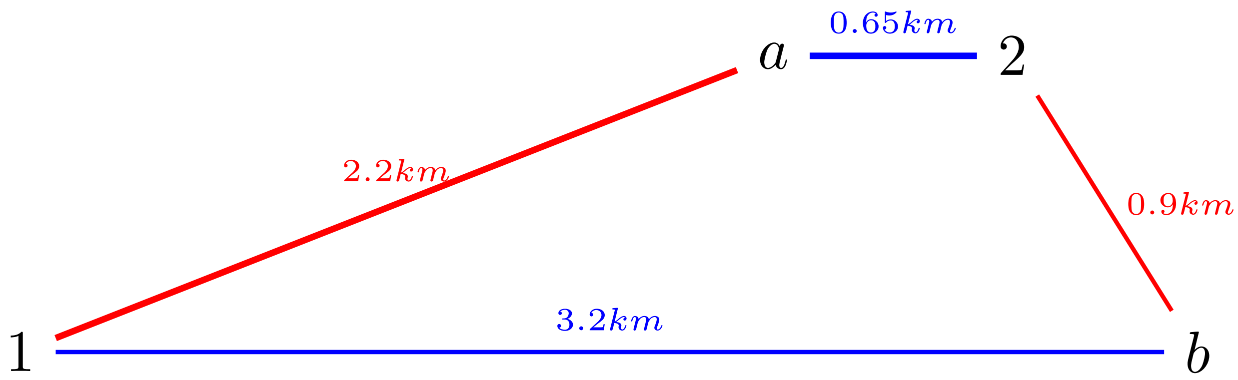
Introduction: motivation in school choice

Tommy Andersson: School choice in Sweden (Matching in Practice website)

Court case in Lund: Schools prioritised children according to walking distances. School a rejected student 1, as all the assigned students lived closer. Student 1 had 2.2 kilometers walking distance to her preferred school a and 3.2 kilometer to her assigned school b. Hence, the walking distance for 1 increased by one kilometer when she was placed at b rather than at a. In contrast, student 2 was assigned to a but would only have had to walk 250 meters further had she been placed at b. The parents argued that student 1 should be given higher priority at a as 1 would lose more from being placed elsewhere. Two courts (Förvaltningsrätten and Kammarrätten) have ruled in favor of the parents.



Introduction: Setting

- Centralized object-allocating institution with **objective function**
- School choice (Abdulkadiroglu and Sönmez, 2003):
  - Diversity (Ehlers et al., 2014; Echenique and Yenmez, 2015; . . .)
  - Boston public schools busing costs 2012: $80 million (Shi, 2016)
  - Trade-off: Remote but highly preferred school
- Refugee allocation (Moraga and Rapoport, 2014; Delacrétaz et al., 2016)
  - Number of assigned refugees (Andersson and Ehlers, 2016)
  - Probability of successful integration
- Social housing; resident allocation (Bronfman et al., 2015); dormitory room allocation (Perach et al., 2008); organ donation

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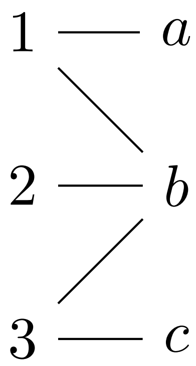
- Richer way of incorporating objective function
- Satisfactory allocation must take preferences into account
- How do we find the Pareto-efficient allocations that are most in line with the planner's objective function?

Model: Basics

- Agents $N = \{i, j, \dots\}$
- Objects $A = \{a, b, \dots\}$
- Acceptability graph $(N \cup A, E)$
- Allocation/matching $x \in X$
- Preferences R_i over $A_i \subseteq A$
- Pareto-improvement: at least one better off, and nobody worse off
- Efficient allocation: no Pareto-improvement possible



R_1	R_2	R_3
a, b	b	b
		c

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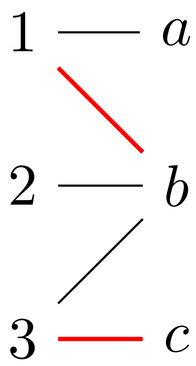

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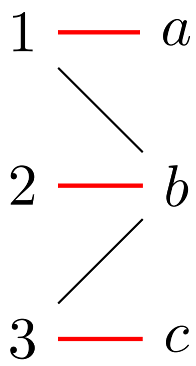

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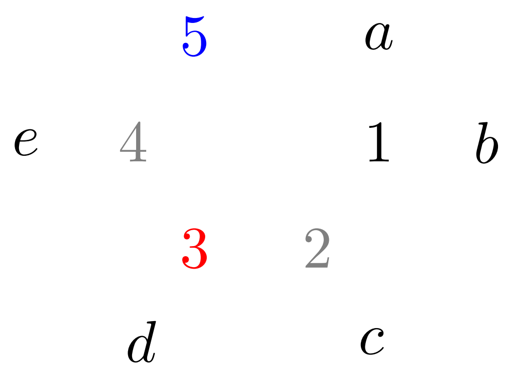
Model: Priorities

- Usually: Schools oversubscribed $\implies$ ration seats $\implies$ priorities
- Absolute proximity: Prioritize students who live closer $\implies$ respecting priorities keeps transportation costs small (but not always)
- Relative proximity: Distance to selected school vs "reference school"
- Ordinal school-priorities cannot fully reflect cardinal information



$R_1, \ldots, R_5$	$\succeq_a$	$\succeq_b$	$\succeq_c$
a	1	1	2
b	5	2	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$

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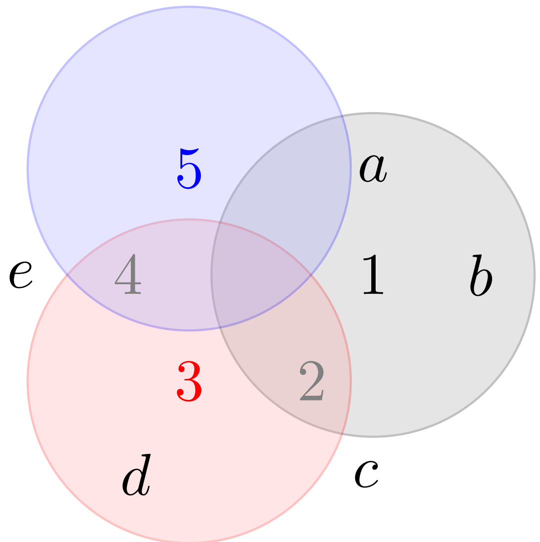

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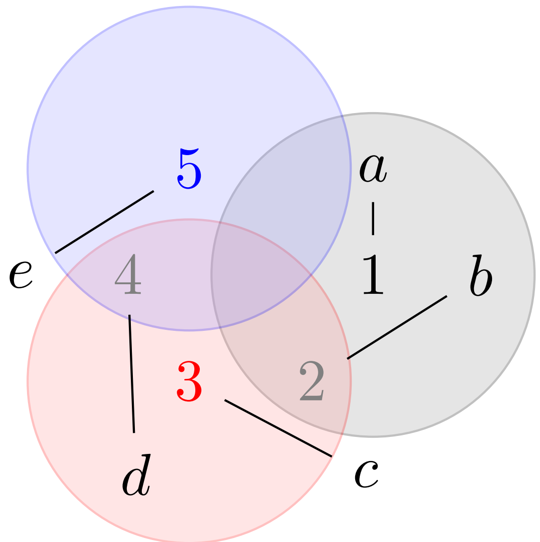

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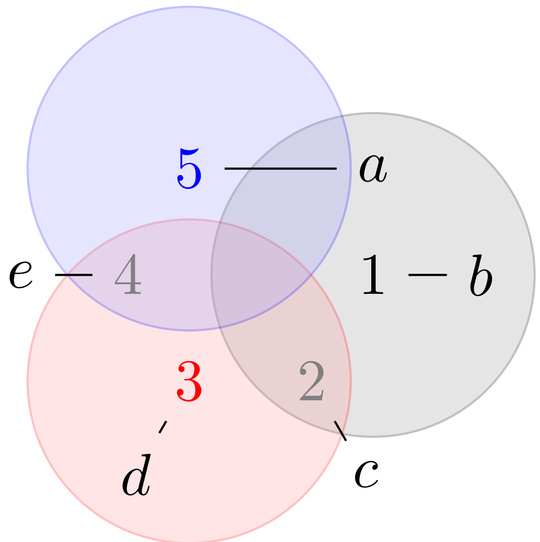

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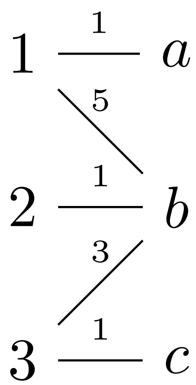
Model: Welfare

- Assigning object a to agent i yields welfare/edge weight $w(i, a) \geq 0$
- Welfare of allocation $x = W(x) =$ sum of edge weights
- Allocation x is **welfare-maximizing** if $W(x) \geq W(y)$ for each $y \in X$
- **Hungarian algorithm** (Kuhn, 1955) finds a welfare-maximizing allocation
- Efficient allocation x is **constrained welfare-maximizing** if $W(x) \geq W(y)$ for each efficient $y \in X$



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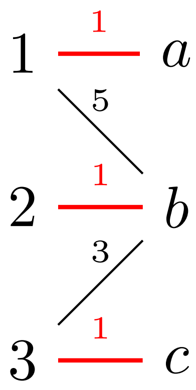

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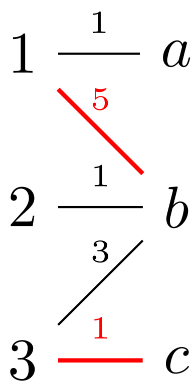

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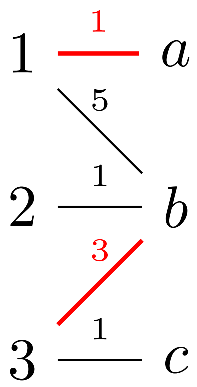

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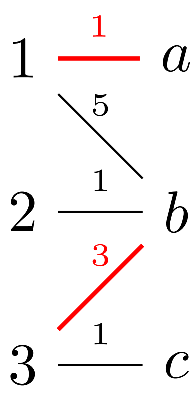
- How do we find a constrained welfare-maximizing allocation?
  Problem considered in this paper: ConstrainedWelfareMax



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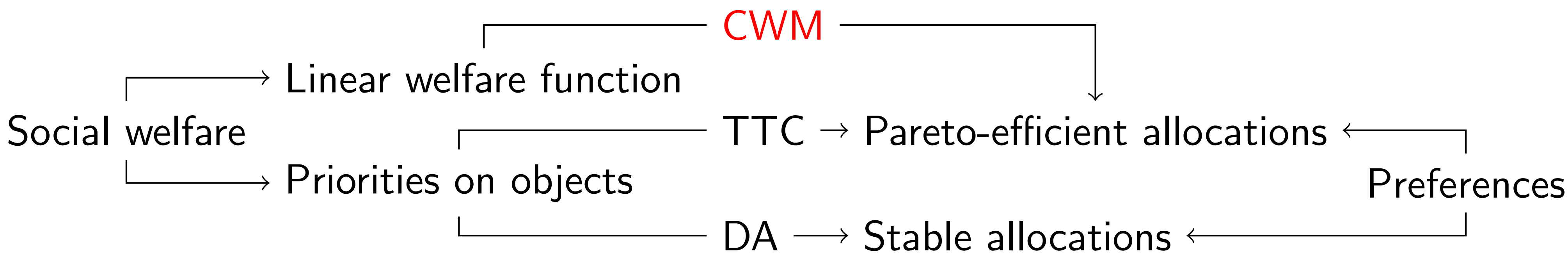
Overview on solution concepts: Objectives and preferences



Special cases: Definitions

Definition (Complete preferences)

For each $i \in N$, $A_i = A$

Definition (Common preference $\succsim$ on A)

For each $i \in N$ and $\{a, b\} \subseteq A_i$, $a \; R_i \; b \iff a \succsim b$ (but A_i can be arbitrary)

Definition (Object-based weights)

For each $\{i, j\} \subseteq N$ and $a \in A_i \cap A_j$, $w(i, a) = w(j, a)$

Definition (Agent-based weights)

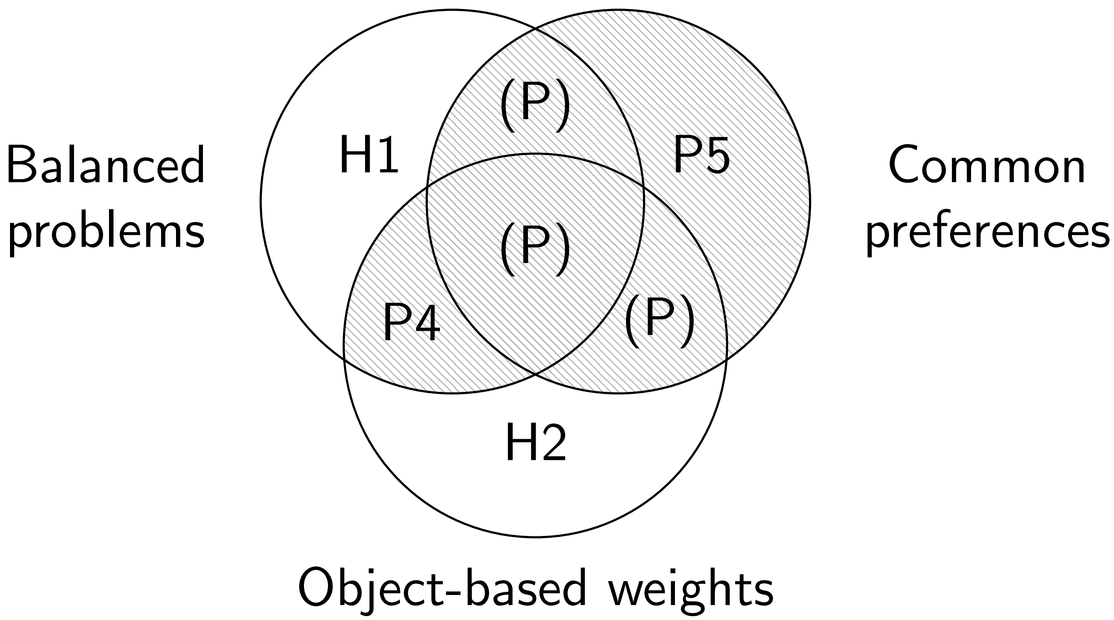
For each $i \in N$ and $\{a, b\} \subseteq A_i$, $w(i, a) = w(i, b)$

Definition (Aligned interests)

For each $i \in N$ and $\{a, b\} \subseteq A_i$, $a \; R_i \; b \iff w(i, a) \geq w(i, b)$

Hardness results: Summary

Complete preferences



H3: For incomplete preferences, the problem is NP-hard even for common preferences, object-based-weights, and balanced problems.
(Note: all the hardness results are for strict preferences, so valid for ties as well)

Hardness results: Theorem 1

Theorem

CONSTRAINEDWELFAREMAX *is NP-hard, even for balanced problems, and complete preferences.*

Proof.

Reduction from SD FEASIBILITY (Saban and Sethuraman, 2015)

- Known: allocation is efficient $\iff$ obtained by serial dictatorship (SD)
- SD FEASIBILITY: The problem of deciding whether a given object a can be obtained by a given agent i in SD.
- Saban-Sethuraman: SD FEASIBILITY is NP-complete, even for balanced problems, and complete preferences
- Add weight 1 for (i, a) and zero for all the other edges...

□

Hardness results: Theorem 2

Theorem

CONSTRAINEDWELFAREMAX *is NP-hard, even for complete preferences, and object-based weights with two values.*

Proof.

Reduction from the problem of finding a minimum size efficient allocation (Abraham et al., 2005)

- add N dummy objects with weight 1, the original objects have weight zero
- put the dummy objects after the originally acceptable objects in the preferences, followed by the originally unacceptable ojects...

□

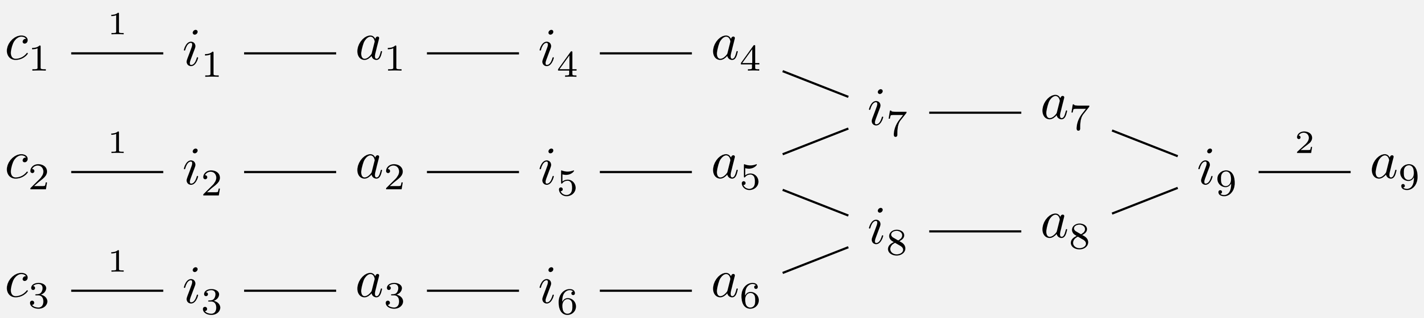
Hardness results: Theorem 3

Theorem

ConstrainedWelfareMax *is NP-hard, even for balanced problems, common preferences, and object-based weights.*

Proof.

Reduction from Exact-3-Cover (Garey and Johnson, 1979), gadget used:



i_1	i_2	i_3	i_4	i_5	i_6	i_7	i_8	i_9
a_1	a_2	a_3	a_1	a_2	a_3	a_4	a_5	a_7
c_1	c_2	c_3	a_4	a_5	a_6	a_5	a_6	a_8
						a_7	a_8	a_9

Hardness results: An implication of Theorem 3

Theorem

SD FEASIBILITY *is NP-complete, even for* **common** *(but incomplete) preferences.*

Proof.

We can extend the construction used in the proof of Theorem 3... □

Easiness results: Complete preferences

Theorem

ConstrainedWelfareMax *is polynomial-time solvable for balanced problems with object-based weights and complete preferences.*

Proof.

By SD... □

Theorem

ConstrainedWelfareMax *is polynomial-time solvable for common and complete preferences.*

Proof.

By the Hungarian method... □

Easiness results: Aligned interests

Definition (Aligned interests)

For each $i \in N$ and $\{a, b\} \subseteq A_i$, $a \, R_i \, b \iff w(i, a) \geq w(i, b)$

- The planner's interests are aligned with the agents'
- Similar condition in Andersson and Ehlers (2016) for refugee allocation: the hosts prefer the larger families
- Efficiency obtained "for free" alongside welfare-maximization

Theorem

Under aligned interests, each welfare-maximizing allocation is efficient

Easiness results: Agent-based weights

Definition (Agent-based weights)

For each $i \in N$ and $\{a, b\} \subseteq A_i$, $w(i, a) = w(i, b)$

- Merit-based priorities: entrance scores in university admissions
- Welfare-maximizing allocation need not be efficient, but
- Weights can be perturbed to get aligned interests:

$$\Delta \equiv W(x) - \max_{y \in X \setminus M(w)} W(y) \text{ (where } M(w) \text{ is the set of WM allocations)}$$

set $\pi(i, a) = w(i, a) + \delta(i, a)$ with $\delta(i, a) \geq 0$ and $\sum_{(i,a) \in E} \delta(i, a) < \Delta$, such that $\delta(i, a) \geq \delta(i, b) \iff a \; R_i \; b$.

Theorem

Under agent-based weights, some welfare-maximizing allocation is efficient, and can be computed as a maximum weight matching.

Easiness results: Object-based weights

Definition (Object-based weights)

For each $\{i, j\} \subseteq N$ and $a \in A_i \cap A_j$, $w(i, a) = w(j, a)$

- Prioritize topics rather than students in college admission
- No conflict between efficiency and allocation size

Theorem

If weights are object-based and positive, then constrained welfare-maximizing allocations are of maximum cardinality

Easiness results: Dichotomous preferences

Theorem

ConstrainedWelfareMax *is polynomial-time solvable for dichotomous preferences.*

Proof.

An allocation is efficient $\iff$ maximum cardinality
so we find a maximum weight maximum cardinality matching by the Hungarian method... $\square$

Integer programming: Competitive equilibrium

Definition (Competitive equilibrium)

Introduce a price $p_a \in \{0, \dots, |A|\}$ for each object a. For an allocation x, (x, p) is a competitive equilibrium if the following conditions hold:

1. Each unassigned object a has price zero: $a \in A \setminus \cup_i x_i \implies p_a = 0$;
2. Each object b preferred to the assigned object a is more expensive: $b \; P_i \; a = x_i \implies p_a < p_b$;
3. Each object b equally good as the assigned object a is no cheaper: $b \; I_i \; a = x_i \implies p_a \leq p_b$.

Theorem (∼Second Welfare Theorem)

Allocation $x \in X$ is Pareto-efficient if and only if there are prices $p \in \{0, \dots, |A|\}^A$ such that (x, p) is a competitive equilibrium.

Integer programming: Variables and objective function

- Choice variables: Assignment $x_{ia} \in \{0,1\}$ and price $p_a \in \{0,\ldots,|A|\}$
- Objective function:

$$W(x) = \sum_{(i,a)\in E} w(i,a)x_{ia}$$

- Simplify formulation: $c_i \equiv \sum_{a\in A_i} x_{ia}$ and $c_a \equiv \sum_{i\in N} x_{ia}$
- Agent assigned a prefers b, $s_{ab} = 1$

$$s_{ab} \equiv \sum_{\substack{i\in N:\\ bP_i a}} x_{ia}$$

- Agent assigned a finds b equally good, $t_{ab} = 1$

Integer programming: Constraints

- Feasibility: $c_i \leq 1$ and $c_a \leq 1$
- Maximality: a is not covered and acceptable to $i \implies i$ must be covered

$$c_i + c_a \geq 1, \quad (i, a) \in E$$

- Object not covered $\implies$ zero price (recall, $0 \leq p_a \leq |A|$)

$$c_a \, |A| \geq p_a$$

- Agent assigned a prefers $b \implies b$ is more expensive

$$(1 - s_{ab})(|A| + 1) + p_b \geq p_a + 1$$

- Agent assigned a finds b equally good $\implies b$ is no cheaper

$$(1 - t_{ab}) \, |A| + p_b \geq p_a$$

A case study: Estonian kindergarten allocation

2016 kindergarten allocation in Harku, Estonia (Veski et al. 2018, JMID)
allocation of 152 children to 155 seats in seven kindergartens

	Stable		Pareto-efficient							
	DA		IA		TTC		CWM		WM	
Rank	Pref.	Dist.	Pref.	Dist.	Pref.	Dist.	Pref.	Dist.	Pref.	Dist.
1	102	103	123	91	102	98	106	102	81	103
2	23	26	6	30	20	25	23	32	31	29
3	5	3	5	3	6	5	4	3	13	7
4	8	7	4	7	11	7	4	6	8	5
5	5	4	6	10	4	4	6	6	6	8
6	7	2	7	5	6	3	8	1	8	0
7	2	7	1	6	3	11	1	2	5	0
Average rank	1.82	1.8	1.61	2.04	1.85	2	1.74	1.64	2.15	1.59
Average distance		3,397		4,035		3,788		3,170		3,023
# blocking agents		0		21		34		26		42
Swaps in post-TTC		2		0		0		0		34

Discussion: Incentives

- Can constrained welfare-maximizing allocations be selected in a strategy-proof way?
- Generally, no
- Object-based weights + complete preferences + at most as many objects as agents $\implies$ Serial dictatorship works
- Relaxing any of these three conditions leaves room for manipulation
- Simple manipulation strategy in school choice: putting a far-away unpopular school as second choice
- Possible solution policy: restricting the choice sets of the parents (i.e. menu-system, like in Boston)

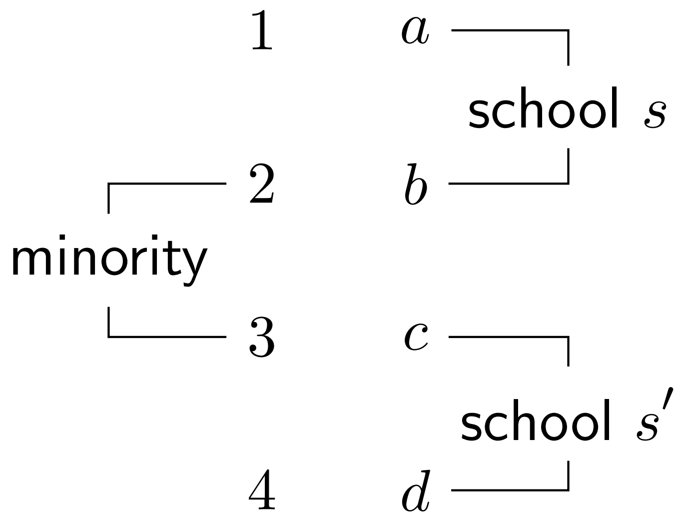
Discussion: Further applications, controlled school choice

1	a
2	b
3	c
4	d

Discussion: Further applications, controlled school choice

1	a	school s
2	b	
3	c	school s'
4	d	

Discussion: Further applications, controlled school choice



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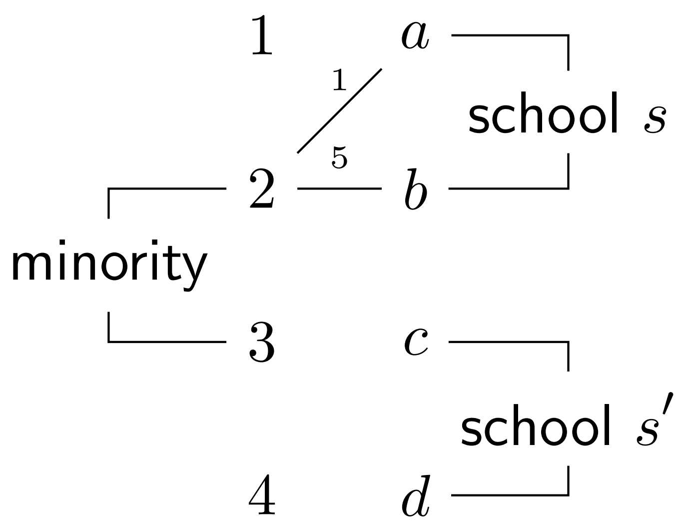

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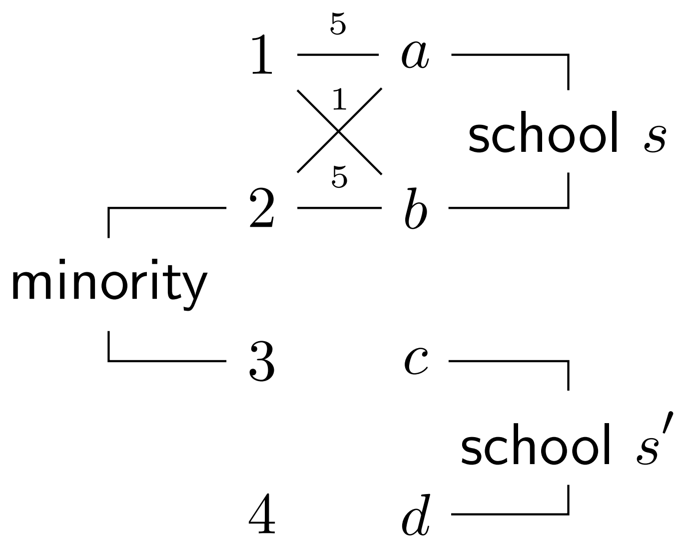

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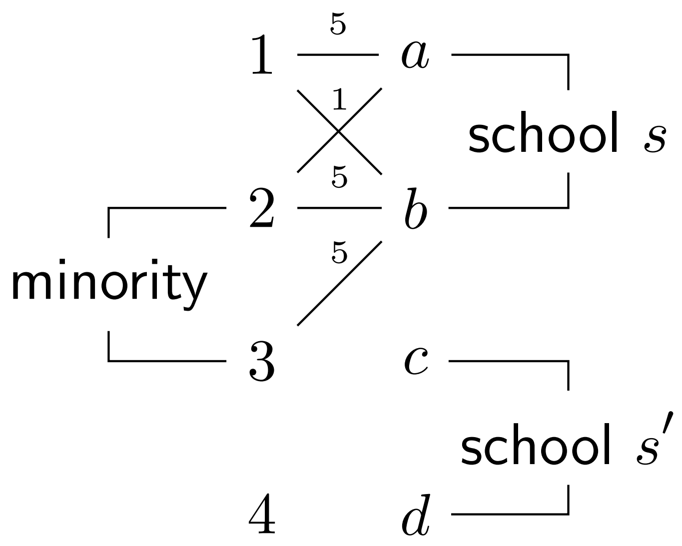

Discussion: Alternative, extended models

- Implicit: Efficiency takes precedence over welfare
  - Alternative: “Most efficient” welfare-maximizing allocation
- Many-to-one or many-to-many assignments (e.g. course allocation)
  - Preference elicitation?
  - Just checking efficiency can be hard
  - Perhaps tractable under some conditions

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Upcoming conferences in Budapest:

- Conference on Economic Design, 12-14 June 2019
  (with a 2-days summer school on computational aspects on 10-11 June)
- 6th World Congress of the Game Theory Society in 2020